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MSC INTERNAL NOTE NO. 65-FM-102

PROJECT APOLLO

ANOTHER METHOD OF SOLVING LAMBERT'S PROBLEM

Prepared by: Francis Johnson, Jr.

(NASA-TM-X-72221) ANOTHER METHOD OF SOLVING LAMBERT'S PROBLEM (NASA) 62 p

N75-70474

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

MANNED SPACECRAFT CENTER

HOUSTON, TEXAS

October 12, 1965

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ANOTHER METHOD OF SOLVING LAMBERT'S PROBLEM

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SUMMARY

What is believed to be a new and unique method of solving Lambert's problem is described herein. The basic idea of this method is to iterate on a single variable through empirical curve fitting. When the two position vectors $\underline{r}_1$ and $\underline{r}_2$ are unequal in magnitude, the variable iterated upon is the true anomaly of the smaller of the two vectors; when $\underline{r}_1$ and $\underline{r}_2$ are equal in magnitude, the variable iterated upon is eccentricity. Eccentric anomaly is nowhere involved in this method and semi-major axis is calculated only as a matter of arbitrary interest after a solution has been found.

The primary feature of this method is that there is absolutely no limitation of the flight angle between $\underline{r}_1$ and $\underline{r}_2$ with which it can cope. Flight angles exactly equal to and nearly equal to 180° can be accommodated with no more difficulty than any other flight angle between 0° and 360°. Multiple orbit solutions (flight angles greater than 360°) are also readily obtained by this method when they exist.

A secondary feature of this method is that it provides the user with a very good sense of the trajectory geometry of solutions, both before and after their actual calculation. It will be obvious to the user why some mathematical solutions are not operationally feasible and why some multiple orbit solutions are non-existent. This user sense of trajectory geometry is attributable to the fact that no new and/or nebulous parameters are involved in this method; all parameters are long-established and have obvious physical significance.

While this method has been extensively programmed in development form, it has not yet been programmed as a subroutine for use in guidance problems. Thus, it has not been compared directly with other methods with regard to numerical results and running time.

OUTLINE OF CONTENTS

1. SUMMARY

2. OUTLINE OF CONTENTS

3. TERMS AND NOTATIONS

   Terms and notations occurring frequently in the text are listed and defined.

4. INTRODUCTION

   The reduction of any realistic three-dimensional Lambert's problem to a two-dimensional problem.

5. BASIC PRINCIPLE OF METHOD - NON-MULTIPLE ORBIT SOLUTIONS WHEN $r_1 \neq r_2$

   The basic principle of the method of finding any solution of any Lambert's problem is derived. This development concerns non-multiple orbit solutions wherein $r_1 \neq r_2$.

6. MULTIPLE ORBIT SOLUTIONS WHEN $r_1 \neq r_2$

   All possible mutliple orbit solutions of any Lambert's problem are derived from the non-multiple orbit solutions described in the preceeding section.

7. LAMBERT'S PROBLEM WHEN $r_1 = r_2$

   All possible solutions, of both non-multiple and multiple orbit types, of the rather specialized category of Lambert's problems wherein $r_1 = r_2$, are developed and defined.

8. CHOICE OF SOLUTIONS

   The problems of applying solutions of a Lambert's problem to a complete flight plan are distinguished from the problem of finding a solution. The characteristics which are necessary in defining different types of solutions are described, such that of any one type, there is never more than one solution to a given problem.

9. FINDING A SPECIFIC TYPE OF SOLUTION

   An iteration method is described which consists of a series of curve fits.

Earlier and future possible methods are described.

APPENDICES

1. Derivation of equation for eccentricity as a function of R, A, and S. Graphic illustration of eccentricity magnitude.
2. Proof that eccentricity is infinite at S1.
3. Derive expression for S2 and S6. Description of factors determining quadrants. How to deal with special case of $X^2 = Z^2$. Complete program logic.
4. Derive expression for S3.
5. Approximate expression for S5.
6. Derivation of value of eccentricity and proof that flight time is zero at S7.
7. Graphic illustrations of the effects of R and A on the magnitudes of S1 to S7.
8. Proof that semi-latus rectum is invariant when A = 180°. Expression for magnitude.
9. Derivation of expression for eccentricity (e_{minp}) corresponding to minimum period when $r_1 = r_2$. Graph as function of A.
10. Elliptical, parabolic, and hyperbolic flight time equations and a specialized series flight time expression.

TERMS AND NOTATIONS

The following are definitions of the terms and notations used herein. These terms and notations are actually defined in the text when first used; they are listed here for the sake of convenient reference.

$\underline{r}_1$ and $\underline{r}_2$	The two position vectors in three-dimensional space involved in a Lambert's problem, their origin being the center of the force field. The convention followed herein in identifying the two vectors is that $r_1 \leq r_2$.
r_1 and r_2	Magnitudes of $\underline{r}_1$ and $\underline{r}_2$, where $r_1 \leq r_2$
R	r_2/r_1 where $r_1 < r_2$
A	The central angle between $\underline{r}_1$ and $\underline{r}_2$, where $0 < A \leq 180^\circ$
Vector Set	The set of elements r_1, r_2, and A, describing the positional geometry of a Lambert's problem when dealt with two-dimensionally in the plane containing $\underline{r}_1$ and $\underline{r}_2$ (or arbitrary plane when $A = 180^\circ$).
S	True anomaly of r_1 on a conic fitting a vector set.
e	Eccentricity of conic fitting a vector set.
P	Period of ellipse fitting a vector set.
p	Semi-latus rectum of a conic fitting a vector set.
X	$R \cdot \sin(A)$
Y	$1 - R \cdot \cos(A)$
Z	$R - 1$
Direct trajectory	That segment of a conic fitting a vector set connecting r_1 and r_2 which subtends the central angle A.
Multiple orbit direct trajectory	Trajectory consisting of the segment of the ellipse fitting a vector set which subtends A (direct trajectory) plus an integer number of complete 2π circuits of the ellipse.
Indirect trajectory	That segment of a conic fitting a vector set connecting r_1 and r_2 which subtends the angle $360^\circ - A$.

Multiple orbit indirect trajectory — Trajectory consisting of the segment of the ellipse fitting a vector set which subtends the angle $360^{\circ} - A$ (indirect trajectory) plus an integer number of complete 2π circuits of the ellipse.

Direct flight time — Flight time between r_1 and r_2 on a direct trajectory. Denoted as FT_d.

Indirect flight time — Flight time between r_1 and r_2 on an indirect trajectory. Denoted as FT_i.

Multiple orbit direct flight time — Flight time between r_1 and r_2 on a multiple orbit direct trajectory. Denoted as FT_{dN}, where (N-1) is the integer number of complete 2π circuits within the elliptical trajectory.

Multiple oribt indirect flight time — Flight time between r_1 and r_2 on a multiple orbit indirect trajectory. Denoted as FT_{iN}, where (N-1) is the integer number of complete 2π circuits within the elliptical trajectory.

INTRODUCTION

Lambert's problem is that of finding a conic trajectory which has a specific flight time between two fixed position vectors. Figure 1 shows two such position vectors, $\underline{r}_1$ and $\underline{r}_2$, as might occur in a typical realistic three-dimensional Lambert's problem.

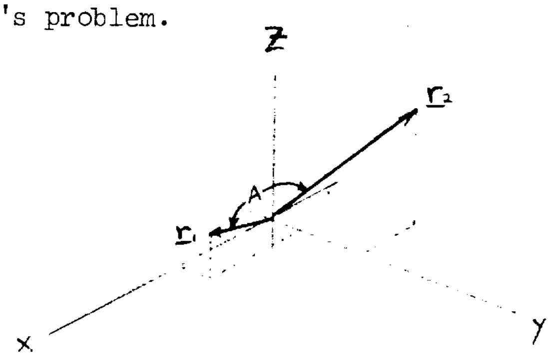

Figure 1

Since any conic connecting the two position vectors in three-dimensional space must lie in the plane containing the two vectors, the problem can be reduced to that of finding the conic trajectory having the desired flight time connecting two radii having magnitudes r_1 and r_2 and having a fixed central angle (A) which is less than or equal to 180°. The quantities, r_1, r_2, and A, in the two-dimensional system shown in figure 2 to which any Lambert's problem can be reduced, will be collectively referred to as a "vector set". The calculation of these quantities is quite straightforward.

$$r_1 = (x_1^2 + y_1^2 + z_1^2)^{\frac{1}{2}}$$

$$r_2 = (x_2^2 + y_2^2 + z_2^2)^{\frac{1}{2}}$$

$$A = \tan^{-1}\left(\frac{|\underline{r}_1 \times \underline{r}_2|}{\underline{r}_1 \cdot \underline{r}_2}\right) \qquad \text{where} \quad 0^\circ < A < 180^\circ$$

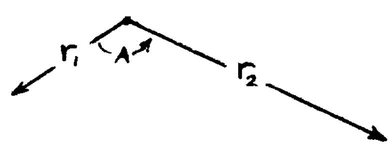

Figure 2

Of course, when A = 180°, no plane is defined by the two position vectors; however, this does not mean that the original problem cannot be reduced to and solved in a two-dimensional system. When A = 180° in a particular Lambert's problem, the trajectory having the desired flight time as derived in a two-dimensional system is free to assume any three-dimensional orientation about the axis formed by $\underline{r}_1$ and $\underline{r}_2$. There will be a "best" three-dimensional orientation of this conic solution from the standpoint of the overall flight plan. Similarly, there will be a "best" solution of any Lambert's problem (it will be shown that there are at least two solutions to any Lambert's problem*), regardless of the value of A, from the same operational standpoint. Both the determination of the best orientation of a solution when $A = 180^{\circ}$ and the determination of the best solution of any problem should not be considered as being part of the Lambert's problem itself, but instead considered operational problems of the overall flight plan of which the Lambert's problem trajectory is but a segment.

PRINCIPLE OF METHOD - NON-MULTIPLE ORBIT SOLUTIONS ($r_1 \neq r_2$)

In the following development pertaining to all problems wherein $r_1 \neq r_2$, the convention is adopted of denoting the smaller of the two vector magnitudes as r_1, the larger as r_2. The side of a typical vector set is depicted such that A is measured counter-clockwise from r_1 to r_2 as shown in figure 3. On the next few pages, a progression of conic trajectories having extreme and significant values of eccentricity and flight time will be fit to the vector set, the periapsis of each trajectory being maintained as horizontal and pointed to the left of the page.

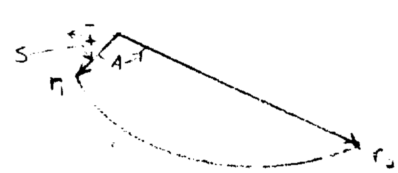

Figure 3 .

*Strictly speaking, except when $A = 180^{\circ}$ and the desired flight time is small enough to preclude multiple orbit solutions.

Only non-multiple orbit trajectories passing directly across A will be initially considered. This type of trajectory will be referred to simply as DIRECT, and a trajectory of this type yielding the desired flight time in a Lambert's problem will be referred to as a DIRECT solution. The true anomaly of r_1 is denoted as S and is measured in a counter-clockwise direction from the periapsis horizontal at the left. The eccentricity of any trajectory fit to a vector set is given by the following equation, the derivation of which is in appendix 1.

$$e = \frac{R - 1}{\cos(S) - R \cdot \cos(S + A)} \qquad \text{where} \quad R = \frac{r_2}{r_1}$$

A theoretical zero flight time would be achieved by a straight-line trajectory directly connecting the two vectors. The value of S for this trajectory is denoted as S1 and is given by the following:

$$S1 = \tan^{-1}\left(\frac{\cos(A) - r_1/r_2}{\sin(A)}\right)$$

$$-90^\circ \leq S1 < 90^\circ$$

The eccentricity of this trajectory is infinite (see appendix 2). The value of S1 as shown in figure 4 is negative. Geometrically defined, S1 is that value of S such that the projections of r_1 and r_2 on the horizontal axis are equal in magnitude and are coincident (have same sign).

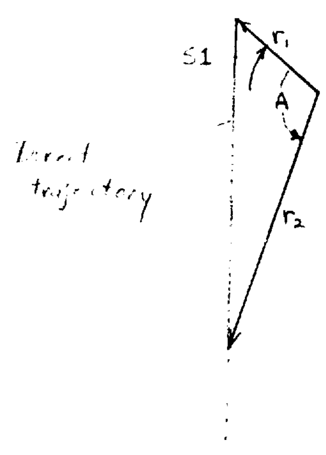

Figure 4

If the vector set is rotated counter-clockwise such that S algebraically increases, the conic trajectories fitting the vector set decrease in eccentricity and increase in flight time. The value of S corresponding to the parabolic trajectory realized when eccentricity decreases to unity, is denoted as S2 and is given by the following:

$$S2 = \tan^{-1}\left(\frac{-xy + z\sqrt{x^2 + y^2 - z^2}}{x^2 - z^2}\right)$$

where $R = r_2/r_1$

$X = R \cdot \sin(A)$

$Y = 1 - R \cdot \cos(A)$

$Z = R - 1$

The factors determining the quadrant in which S2 lies are described in appendix 3, together with a derivation of the above expression for S2. As pictured in figure 5, S2 is negative.

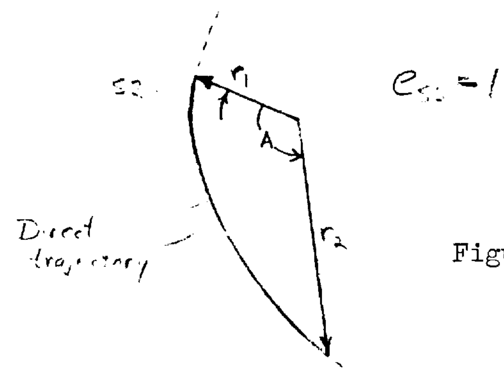

Figure 5

Rotating the vector set further counter-clockwise, the flight time of the fitting trajectories increases further and eccentricity decreases until it reaches a minimum. The value of S corresponding to the trajectory having the minimum possible eccentricity is denoted as S3 and is given by the following:

$$S3 = \tan^{-1}\left(\frac{\sin (A)}{r_1/r_2 - \cos (A)}\right) = \tan^{-1}\left(\frac{X}{Y}\right) = S1 + 90^\circ = \frac{S2 + S6}{2}$$

$$0^\circ \leq S3 < 180^\circ$$

The derivation of this first expression is given in appendix 4. As shown in figure 6, S3 is that value of S such that the projections of r_1 and r_2, on an axis perpendicular to the periapsis horizontal, are equal in magnitude and are coincident. The fact that S3 is exactly half way between S2 and S6 in all problems, as indicated in the last expression above, is based on computer programming experience, not on mathematical derivation.

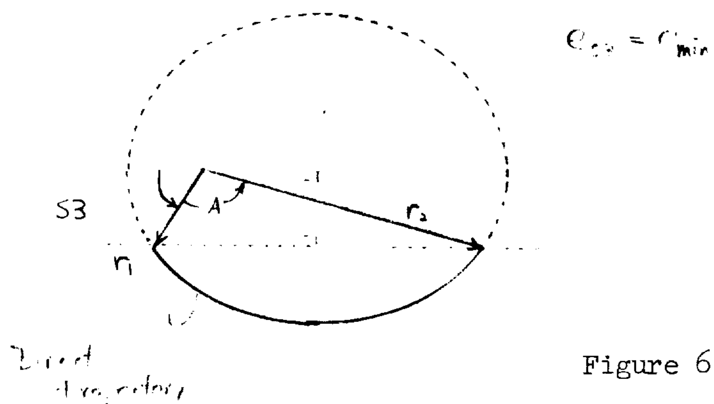

Figure 6

Rotating the vector set further counter-clockwise until r_2 becomes the periapsis of the fitting ellipse as shown in figure 7, eccentricity increases from its minimum of e_{S3} and flight time continues to increase. The value of S at this condition is denoted as S4 and is given by the following:

$$S4 = 180^{\circ} - A$$

The values of eccentricity and flight time at S4 are neither extreme nor significant; S4 is introduced only for convenience in dealing with multiple-orbit solutions later.

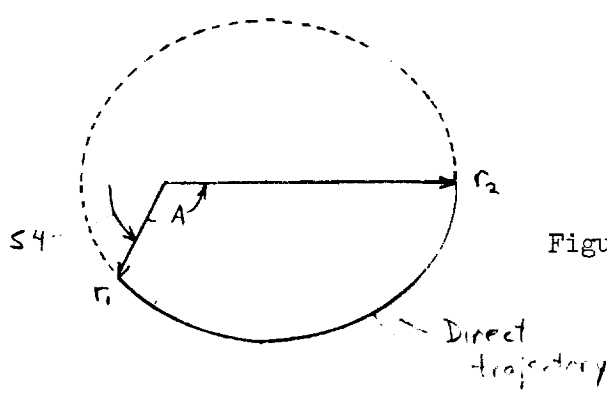

Figure 7

As the vector set continues to rotate counter-clockwise, both the eccentricity and flight time continue to increase while the period of the ellipse fitting the vector set decreases to a minimum. The value of S corresponding to this minimum-period ellipse is denoted as S5. While exact explicit expressions for all other significant values of S have been readily derived, no such expression has been derived for S5. A more detailed discussion of S5 and an approximate empirical expression for its value are given in appendix 5. The values of eccentricity and flight time at S5 are neither extreme nor significant.

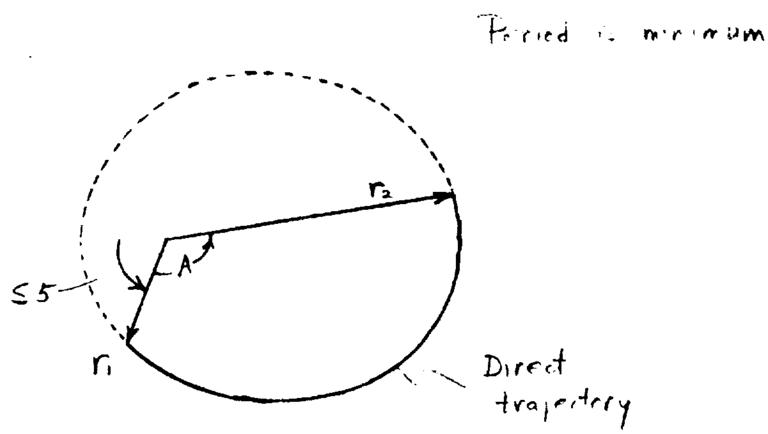

Figure 8

Continuing to rotate the vector set further counter-clockwise, eccentricity increases to unity and flight time becomes infinite. The value of S corresponding to this limiting parabolic trajectory is denoted as S6. The expression for S6 is very similar to that for S2, their both being the two roots of the same quadratic equation. See appendix 3 for a description of the factors determining the quadrant in which S6 lies and a derivation of the expression for S6.

$$S6 = \tan^{-1}\left(\frac{-X\cdot Y - Z\sqrt{X^2 + Y^2 - Z^2}}{X^2 - Z^2}\right)$$

where $R = r_2/r_1$

$X = R\cdot\sin(A)$

$Y = 1 - R\cdot\cos(A)$

$Z = R - 1$

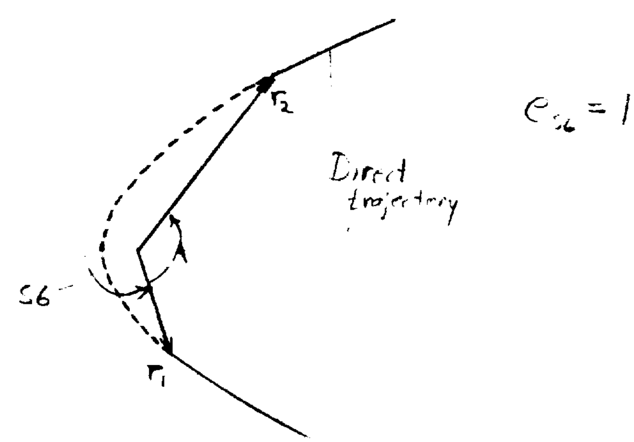

Figure 9

The preceeding series of from S1 to S6 has shown that the flight time on all possible non-multiple orbit trajectories directly across A can be represented as a continuous, single-valued, monotonic function of S over the range of from S1 to S6. The range of flight time represented, as shown in figure 10, is from zero to infinity.

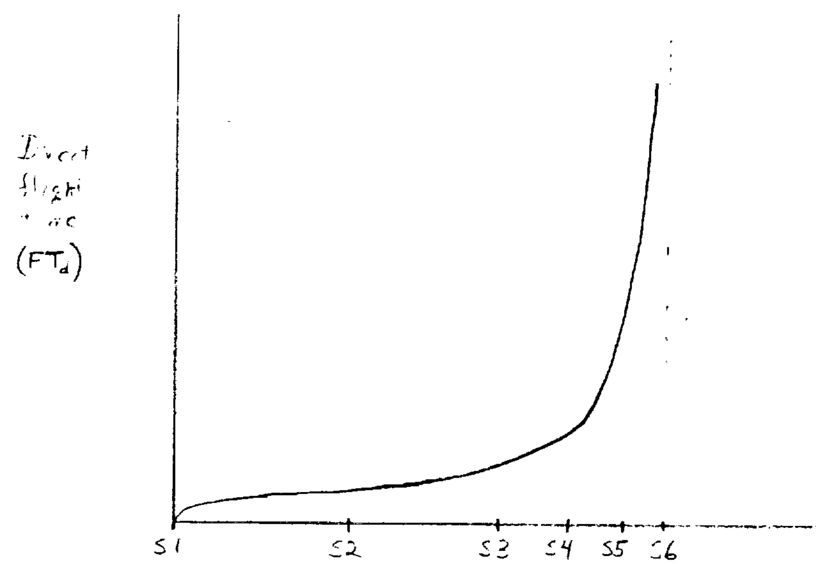

Figure 10

It must be said at this point that the assertion that direct flight time expressed as a function of S has no maxima, minima, or inflection points between S1 and S6, is based entirely upon early intuition and later experience. An expression for the derivative of direct flight time with respect to S for any vector set has yet to be derived and proven to have no zero values between S1 and S6. All programming experience to date has indicated that no such zero value of this derivative exists.

All of the trajectories represented in figure 10 are direct; i.e., they all pass directly across A without ever going around the other side of the center of attraction. As might be expected, there is another continuum of trajectories passing around the other side of the center of attraction having flight times varying from zero to infinity. This type of non-multiple orbit trajectory, which passes around the "back" side of the center of attraction without ever passing across A, is referred to as INDIRECT.

For a given value of S, there is only one conic which fits a vector set. When the conic is an ellipse, that segment of it subtending A constitutes a direct trajectory while the remainder of the ellipse is an indirect trajectory. Thus, whenever the value of S is such that $e < 1$, there will be both a direct and an indirect trajectory fitting a vector set, the direct and indirect flight times always being different except in singular cases.

Only when S is between S2 and S6 is the conic fitting the vector set an ellipse; i.e., the S of any ellipse fitting the vector set must lie between S2 and S6. The lower limit of the range of S associated with the indirect trajectories fitting a vector set is S2. By definition, the conic fitting the vector set at S2 is a parabola which subtends A. Referring to figure 5, it can be seen that the indirect flight time at S2 is infinite. As S increases from S2 to S6, indirect flight time decreases, as can be seen in figures 5 through 9, wherein the indirect trajectories are represented by the dashed portions of the fitting conics.

The upper limit of the range of S associated with indirect trajectories is denoted as S7. Continuing the progressive counter-clockwise rotation of a typical vector set as shown in figures 4 through 9, S7 corresponds to a theoretical zero indirect flight time. Its value and that of the corresponding eccentricity are given as follows. See appendix 6 for complete derivations.

$$S7 = 180^{\circ} - A/2$$

$$e_{S7} = \sec(A/2)$$

This indirect trajectory is not readily depicted in figure 11 as it is coincident with both r_1 and r_2 and passes through the center of attraction.

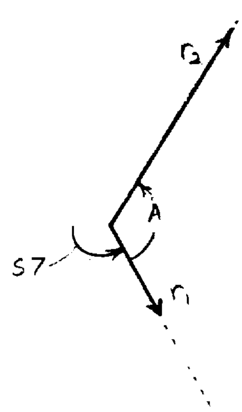

Figure 11

Figures 12 and 13 show the continuous variations of direct and indirect flight times and eccentricity as a function of S between S1 and S7 for a typical vector set. The following table summarizes values of S corresponding to significant values of eccentricity and flight time.

S	e	Direct Flight Time (FT_d)	Indirect Flight Time (FT_i)
S1	∞	0	*
S2	1	finite	∞
S3	e_{min}	finite	finite
S6	1	∞	finite
S7	sec(A/2)	*	0

* Not applicable

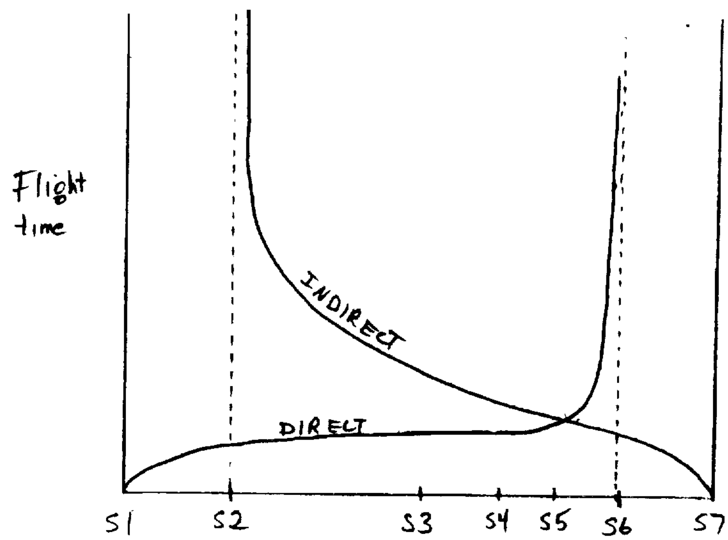

Figure 12

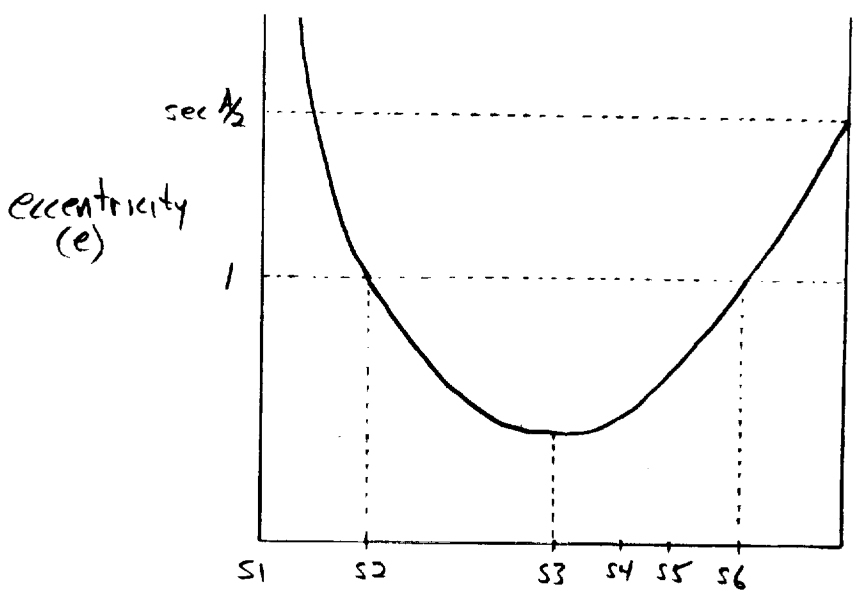

Figure 13

As is the case with direct flight time, an expression for the derivative of indirect flight time with respect to S for any vector set is yet to be derived and proven to have no zero values between S2 and S7. The assertion here that indirect flight time is a monotonically decreasing function of S is based upon programming experience.

There is a relationship between S and direct and indirect flight times similar to that shown in figure 12 for any vector set wherein $r_1 \neq r_2$. Thus, for any specific finite flight time greater than zero, there are two non-multiple orbit solutions for any vector set, the one solution being a direct trajectory and the other an indirect trajectory.*

The magnitudes of the seven significant values of S of from S1 to S7 are determined by the A and the ratio r_2/r_1 of the vector set. These relationships are graphically illustrated in appendix 7. Significant invariant relationships are,

$$S3 = S1 + 90^{\circ} \qquad\qquad S3 = (S2 + S6) / 2$$

It can be seen in figure 12 that where the two curves cross, direct flight time is equal to indirect flight time. Thus, for any vector set, there is a fitting ellipse upon which the flight time around one side of the center of attraction is equal to the flight time around the other side; the value of S corresponding to this ellipse is between S5 and S6, usually very close to S5 for most vector sets.

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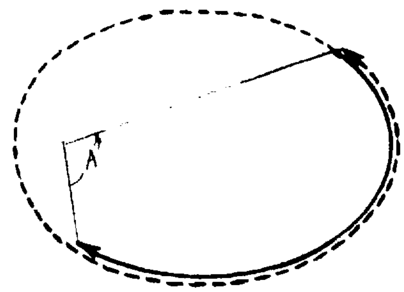
* When $A = 180^{\circ}$, these two solutions for a given flight time are geometrically identical and thus it cannot be said that there are two different solutions.

When the A of a vector set is 180°, the terms "direct" and "indirect", as previously defined, are not applicable. In this instance, and only in this instance, would the relationship between flight times and S as shown in figure 12 be absolutely symmetrical about S4; the value of S4 then being zero.

MULTIPLE ORBIT SOLUTIONS ($r_1 \neq r_2$)

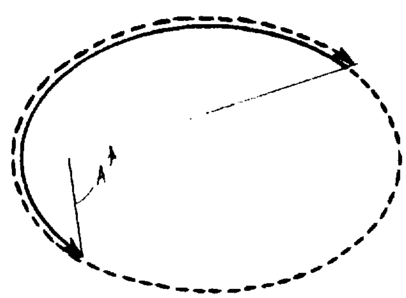
The notations FT_{dN} and FT_{iN} have been found convenient in dealing with multiple orbit solutions to Lambert's problem. They denote, respectively, direct and indirect flight time on orbit number N, where N is an integer greater than unity. When N is not expressed, it is understood to be unity. The notations FT_d and FT_i thus denote direct and indirect non-multiple orbit flight times, respectively, as described up to this point on the preceeding pages.

Figure 14 shows the physical significance of direct and indirect multiple orbit flight times on the same ellipse fitting a vector set; S is the same for both trajectories. The darker solid lines in figure 14 represent the basic direct and indirect trajectories and flight times, the dashed lines represent the additional elliptical periods. A convenient interpretation of the notations FT_{dN} and FT_{iN} is to consider the alphabetic subscripts (d) or (i) as denoting which part of the ellipse, direct or indirect, is traversed N times by the trajectory.



Direct flight time on second orbit (FT_{d2})

Figure 14



Indirect flight time on second orbit (FT_{i2})

The period (P) of an ellipse fitting a vector set for any value of S between S2 and S6 is equivalent to the sum of direct and indirect flight times.

$$P(S) = FT_d(S) + FT_i(S)$$

Figure 15 shows the variation of period as a function of S for a typical vector set. As noted earlier, an exact explicit expression for S5, whereat P is a minimum, has not yet been derived; the empirical expression for S5 in appendix 5, which is based upon computer results, is approximate.

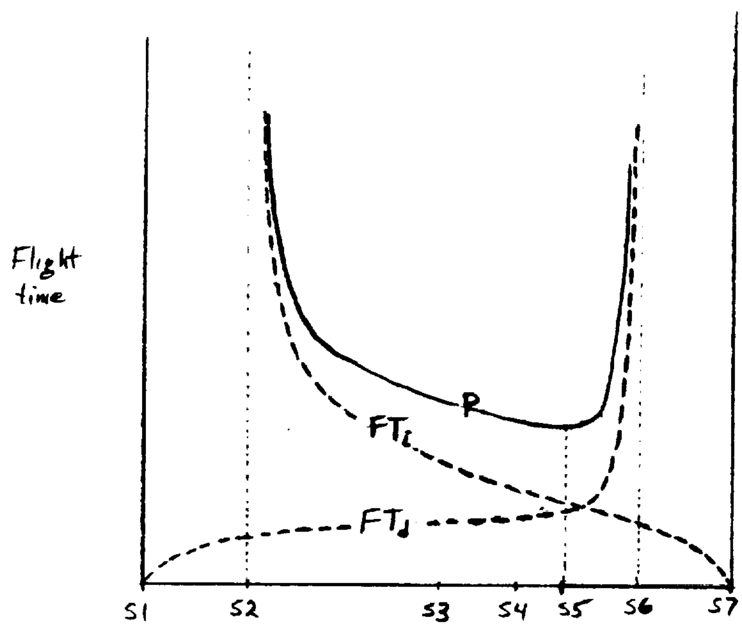


Figure 15

S5 is <u>not</u> equivalent to the value of S where $FT_d = FT_i$, except in singular cases. Taking the derivative of the last equation with respect to S, it can be seen that S5 occurs where the sum of the derivatives of direct and indirect flight time is zero.

At S5, where P is minimum:
$$\frac{dP}{dS} = \frac{dFT_d}{dS} + \frac{dFT_i}{dS} = 0$$

At any specific value of S between S2 and S6, direct and indirect multiple orbit flight times are given by the following expressions:

$$FT_{dN} = FT_d + (N-1) \times P = N \times FT_d + (N-1) \times FT_i$$

$$FT_{iN} = FT_i + (N-1) \times P = (N-1) \times FT_d + N \times FT_i$$

Figure 16 shows the relationships between the basic direct and indirect flight times (FT_d and FT_i), the second orbit direct and indirect flight times (FT_{d2} and FT_{i2}), and S of a typical vector set.

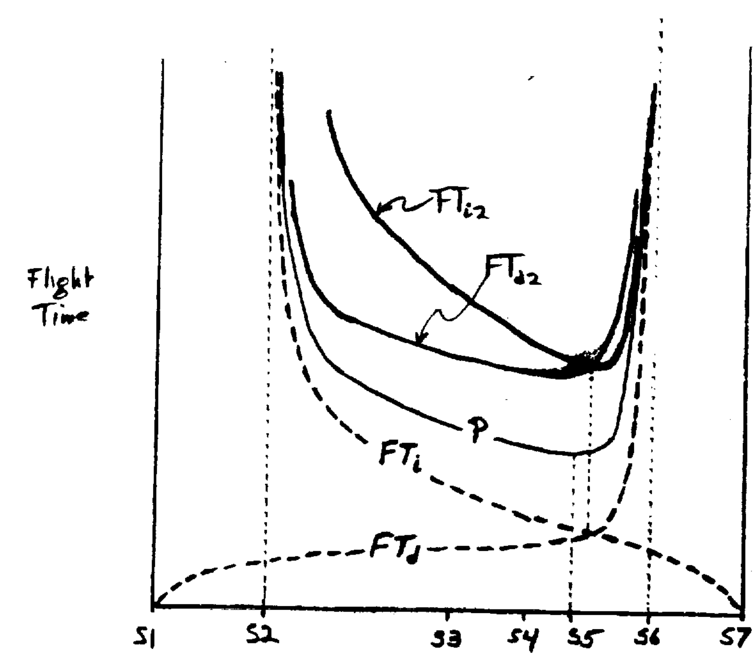


Figure 16

The minimum values of multiple orbit direct and indirect flight times occur at the values of S where the derivatives of the above equations with respect to S equal zero.

At minimum FT_{dN}: $$\frac{dFT_{dN}}{dS} = \frac{dFT_d}{dS} + (N-1)\,\frac{dP}{dS} = N \cdot \frac{dFT_d}{dS} + (N-1)\,\frac{dFT_i}{dS} = 0$$

$$\text{Therefore, } \left(\frac{N}{N-1}\right) \frac{dFT_d}{dS} = - \frac{dFT_i}{dS}$$

At minimum FT_{iN}: $$\frac{dFT_{iN}}{dS} = \frac{dFT_i}{dS} + (N-1)\,\frac{dP}{dS} = (N-1)\,\frac{dFT_d}{dS} + N \cdot \frac{dFT_i}{dS} = 0$$

$$\text{Therefore, } \left(\frac{N}{N-1}\right) \frac{dFT_i}{dS} = - \frac{dFT_d}{dS}$$

Since dFT_d/dS is always positive and dFT_i/dS is always negative, in order for these last equations to be satisfied, the minimum of FT_{dN} must always occur at a value of S where dP/dS is negative and the minimum of FT_{iN} must always occur at a value of S where dP/dS is positive. In other words, the minimum values of FT_{dN} of a vector set all occur at values of S less than S5, while the minimum values of FT_{iN} all occur at values of S greater than S5. This effect can be seen in figure 16.

As N becomes very large, the coefficient N/(N-1) in the above equations approaches unity, in which limiting case the equations are satisfied by the condition

$$\frac{dFT_d}{dS} = \frac{-dFT_i}{dS}$$

which occurs at S5. In other words, as N becomes very large, the S corresponding to a minimum flight time approaches S5 as a limit.

LAMBERT'S PROBLEM WHEREIN $r_1 = r_2$

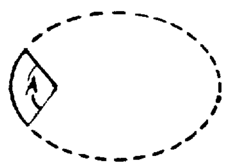
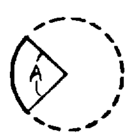
It has been shown on the preceeding pages that S can be conveniently used as the single independent variable in the iterative process of solving Lambert's problem wherein $r_1 \neq r_2$; more specifically, the relationships between S and flight times lend themselves quite readily to empirical curve fitting in the iterative process of finding the value of S yielding a desired value of a specific type of flight time (the value of S having been determined, the conic fitting the vector set is defined and a solution to Lambert's problem is achieved). There are several features of these relationships which make S a very convenient independent variable. The range of S associated with all possible values of any specific type of flight time is finite and its limits are explicitly defined. Within any such range of S, flight time is a single-valued, continuous, smooth, function of S. FT_d is absolutely continuous and smooth at S2 as is FT_i at S6; i.e., in no case does the relationship between S and non-multiple orbit flight time experience a discontinuity or exhibit any extraordinary behavior when the conic fitting the vector set becomes a parabola or changes from an ellipse to a hyperbola or vice versa.

When the two position vectors in a Lambert's problem are equal in magnitude, the parameter S cannot be used as the independent variable in the iterative process of finding a solution. This is because when $r_1 = r_2$, the true anomalies of the position vectors are not continuously variable; the true anomaly of a position vector can have one of two magnitudes having arbitrary sign. This is due to the inherent symmetry of any conic about its major axis.

When any conic is fit to a vector set having $r_1 = r_2$, the true anomaly of one position vector always equals the negative of the true anomaly of the other.

As in the case of vector sets wherein $r_1 \neq r_2$, there is a sequence of conics which can be fit to vector sets wherein $r_1 = r_2$, which illustrates a continuous variation of both FT_d and FT_i of from zero to infinity. The sequence of conics for vector sets having $r_1 \neq r_2$ was illustrated in figures 4 through 9 and 11. The sequence of conics for vector sets having $r_1 = r_2$ is shown in the following table. The terms "direct" and "indirect" apply to vector sets wherein $r_1 = r_2$ equally as well as to the much more general category of vector sets wherein $r_1 \neq r_2$.

Sequence of conics fit to a typical vector set having $r_1 = r_2$

Significance in Figure 17	Conic fit to vector set (Direct trajectory ———, Indirect trajectory - - - - -)	Eccentricity of Conic	FT_d	FT_i	True Anomaly	P/r
a		∞	0	*	$\pm A/2$	∞ (* when $A = 180°$)
b		1	finite	∞	$\pm A/2$	$1 + \cos\left(\frac{A}{2}\right)$
c		$0 < e < 1$	finite	$<$ finite	$\pm A/2$	$1+\cos(A/2) > P/r > 1$
d		0	finite	$<$ finite	*	1
e		$0 < e < 1$	finite	finite	$\pm\left(180° - \frac{A}{2}\right)$	$1 > P/r > 1 - \cos(A/2)$
e'		$e_{minp} = \sec\left(\frac{A}{2}\right) - \tan\left(\frac{A}{2}\right)$	finite	finite	$\pm(180° - A/2)$	$\sin(A/2)$
f		1	∞	finite	$\pm\left(180° - \frac{A}{2}\right)$	$1 - \cos(A/2)$
g		$\sec(A/2)$	*	0	$\pm(180° - A/2)$	0 (* when $A = 180°$)

* Undefined or not applicable

Since S is not continuously variable when $r_1 = r_2$, it is necessary to use a different parameter as the independent variable in the iterative process of solving this type of Lambert's problem. The conic parameters, semi-major axis, semi-minor axis, and apiapsis radius, are undesirable for this purpose because of their behavior when the conic becomes parabolic. The conic parameters semi-latus rectum, periapsis radius, and eccentricity are much more suitable since they do not exhibit such behavior.

Referring to the preceeding table, it appears that both FT_d and FT_i can be expressed as single-valued, continuous, monotonic functions of semi-latus rectum (p), or for the sake of generality, of the ratio p/r. It would thus seem that semi-latus rectum is a suitable independent variable in the iterative process. However, in vector sets wherein $A = 180^\circ$, regardless of whether r_1 is equal to r_2 or not, semi-latus rectum magnitude is invariant, being a function only of the ratio r_2/r_1. See appendix 8. It can be seen that the expressions for the ratio p/r in the preceeding table become equal to 1 when $A = 180^\circ$. When $r_1 = r_2$ and A is very close to 180°, the range of p/r associated with all ellipses, the two parabolas, and low eccentricity hyperbolas fitting the vector set, is very restricted; as A approaches 180°, this range of p/r becomes increasingly restricted until when $A = 180^\circ$, p/r becomes invariantly equal to 1. Because of this restriction and possible invariance of p/r in this realm of operationally practical trajectories, p/r is considered unsuitable as the independent variable in the iteration process of solving Lambert's problem when $r_1 = r_2$.

Periapsis radius and eccentricity have the disadvantage that both FT_d and FT_i are double-valued functions of them over part of their ranges.*
Of the two, it is felt that eccentricity is the more suitable independent variable because the only variable in the flight time equations used in this type of problem is eccentricity, and the relationships between eccentricity and flight times are very similar to those between S and flight times. This latter similarity can be seen in figure 17 (compare to figure 15), where the abscissa has been made double-valued to eliminate the otherwise graphic confusion of both FT_d and FT_i being double-valued functions of eccentricity.

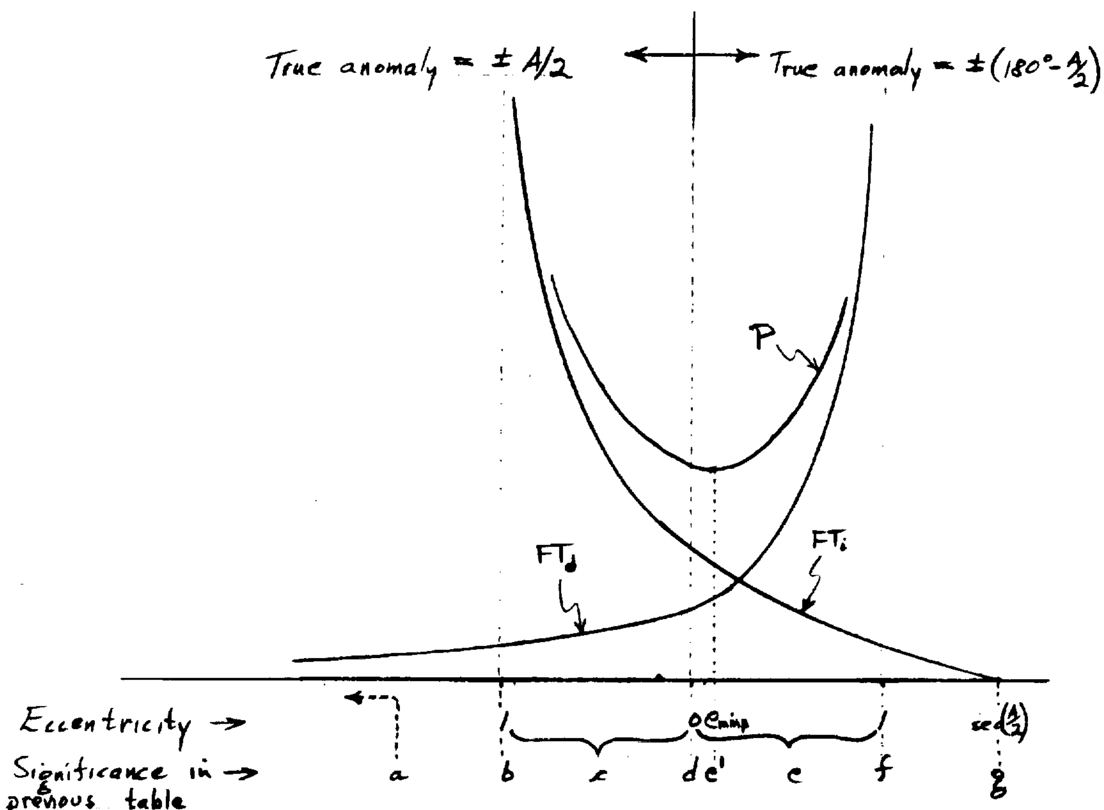


Figure 17

* It should be pointed out that both FT_d and FT_i can be expressed as continuous, smooth, single-valued, monotonically increasing functions of the apsis radius intersecting that segment of the conic which constitutes the trajectory concerned.

Any specific point on the abscissa scale in figure 17 represents a specific conic fit to the vector set. Significant points and ranges along this abscissa scale are alphabetically identified for relation to the sequence of conics illustrated in the preceeding table. Since this abscissa scale of eccentricity is double-valued, when the value of eccentricity is less than sec A/2, in order to define a specific conic it is necessary to specify not only its eccentricity but also whether the true anomalies are $\pm$ A/2 or $\pm$ (180°- A/2).

Because of the very close similarity between figures 15 and 17, it would seem to be rather repetitious at this point to go into any analysis of multiple orbit flight times for vector sets wherein $r_1 = r_2$. Curves representing multiple orbit flight times similar to these shown in figure 16 can be similarly derived from figure 17 by the addition of values of period (P) and FT_d or FT_i. The analysis of minimum multiple orbit flight times for vector sets having $r_1 \neq r_2$ presented on pages 24 and 25 can be similarly applied to those having $r_1 = r_2$, using a double-valued abscissa scale of eccentricity instead of S. The similarity between the roles of S5 when $r_1 \neq r_2$ and e_{minp} when $r_1 = r_2$ with regard to minimum multiple orbit flight times is rather straightforward. However, two significant differences between the two categories of Lambert's problems, those having $r_1 \neq r_2$ and those having $r_1 = r_2$, should be pointed out. The range of eccentricity associated with the complete range of FT_d of from zero to infinity, is not closed, whereas the similarly associated range of S when $r_1 \neq r_2$ is closed (S1 to S6). There is an exact explicit expression for the value of eccentricity (e_{minp}) of the ellipse fitting the vector set having minimum period (the true anomalies on this ellipse always being $\pm$ (180° - A/2)). This expression and its derivation are given in appendix 9.

There is no known exact explicit expression for S5; see appendix 5. The close similarity between the relationships of S to flight times when $r_1 \neq r_2$ and eccentricity to flight times when $r_1 = r_2$ is not only convenient to an understanding, but it also facilitates programming logic in the empirical curve fitting iterative process of finding solutions to both types of Lambert's problems.

CHOICE OF SOLUTIONS

The preceeding pages have been a description of different types of solutions of Lambert's problem, the problem being that of defining a conic trajectory having a specified flight time between two position vectors. It has been shown that for any value of required flight time there are at least two solutions for any vector set and additional solutions of the multiple orbit type if the required flight time is large enough.

A set of solutions of a given Lambert's problem, as derived from relationships of the type shown in figures 15 and 17, are completely insensitive to physical reality and to the remainder of the vehicle's flight plan. Some of these solutions could well consist of trajectories having periapsis radii which are less than the physical radius of the central attracting body. Similarly, these solutions are not concerned with how the vehicle arrived at the initial position vector in the problem or with what the vehicle has to do after it reaches the terminal position vector.

Of a group of solutions to a Lambert's problem, there is usually one which is the best from the standpoint of its relationship to the immediate flight plan and physical reality. The obviously foolproof method of finding this one best solution is to find all possible solutions to the Lambert's problem and analyze them. However, this procedure is not efficient and in most practical operational applications is unnecessary. The nature of the immediate flight plan, linked to the Lambert's problem trajectory at its initial and terminal position vectors, will usually indicate either which solution is the best or which category of solutions contains the best. For instance, the angular sense of motion of the flight plan, in relation to the vector set of the problem and the center of the local force field, will usually determine whether the best solution is direct or indirect, and the magnitude of the flight time required will usually indicate, based upon experience, whether the best solution is of the multiple or non-multiple orbit type. A good example is in the solving of Lambert's problem in thrusting programs where one need only be concerned with direct non-multiple orbit solutions; the only applicable solution is always on the FT_d curve.

When the possibility exists of the best solution of a Lambert's problem in an operational application being of the multiple orbit type, it would appear to be necessary to find both solutions on either the FT_{dN} or FT_{iN} curve involved and analyze them to determine which is the more operationally suitable. This is because at present no simple generalities have been established as to how two solutions on a given curve differ with regard to trajectory characteristics. The two solutions on a curve are presently differentiated only by stating whether their S (or e when $r_1 = r_2$) is greater or less than that of the minimum flight time on the curve. It is hoped that experience will reveal some relationship between the values of S (and eccentricity when $r_1 = r_2$) of these pairs of solutions and their energies or other characteristics influencing operational suitability.

Regardless of whether it is obvious which of the types of flight times is going to be the best operational solution of Lambert's problem before finding any of these solutions, or if the factors determining the suitability of a solution are not as obvious such that a number of solutions must be found and analyzed, it is necessary in finding any one solution to constrain the iterative process to a specific type of flight time.

To effectively perform this constraint, it is necessary to be able to distinguish between the different types of flight time solutions to Lambert's problem. The two most general categories of flight time appear to be multiple orbit and non-multiple orbit. Within these two categories, specifying the following characteristics of a flight time will define it as a unique solution.

Non-multiple orbit - 1. Direct or indirect

Multiple orbit - 1. Direct or indirect
2. Number of orbit
3. S greater or less than that of minimum flight time when $r_1 \neq r_2$, or eccentricity greater or less than that of minimum flight time when $r_1 = r_2$.

FINDING A SPECIFIC TYPE OF SOLUTION

The iterative process of finding any one type of solution of a Lambert's problem involves the calculation of values of flight time for different values of S when $r_1 \neq r_2$ and for different values of eccentricity when $r_1 = r_2$. These calculations of flight time are normally based upon established explicit equations of the form,

$$FT = FT\ (\theta,\ r,\ e)$$

where θ is the true anomaly at position radius r on a conic having eccentricity e, and FT is the flight time from periapsis. Of course, the gravitational constant μ of the central force field occurs in these equations but its value is a constant in any one problem. When $r_1 \neq r_2$, both eccentricity and the two values of θ in a problem (of r_1 and r_2) are explicit functions of S; when $r_1 = r_2$, θ can have either of two fixed values within a problem and eccentricity is the independent iterative variable.

There is no one explicit flight time equation based on true anomaly which is applicable to all values of eccentricity; three different equations must be used to accommodate all possible values. Each equation is applicable to values of eccentricity either less than, equal to, or greater than unity. These equations are given in appendix 10. The inherent practical limit of the continuity of these elliptical and hyperbolic flight time equations in the realm of eccentricities very close to unity in a given Lambert's problem, is also discussed in appendix 10 together with series expressions for flight time which do not exhibit this undesirable behavior.

The flight time to be calculated is not always found by simply taking the difference between the flight times from periapsis to the two position vectors as given directly by these equations or series. It is necessary to consider whether the flight time to be calculated is direct or indirect and interpret the signs of the values given by the equations or series accordingly. Multiple orbit flight times are calculated by adding appropriate integer multiples of period to either the direct or indirect non-multiple orbit flight times.

There are a number of ways of controlling the iterative process in finding a solution of a given type. When this method of solving Lambert's problem was first developed in 1961, the iterative procedure used was to vary S by discrete steps, changing the sign and size of the steps such that the corresponding flight time converged on the desired value. This simplest type of iterative procedure is very time consuming.

The method presently used by the author consists of a series of fits of equilateral hyperbolas to known points on the appropriate flight time-S curve (or flight time-eccentricity curve when $r_1 = r_2$)*. The analytic form of this hyperbola is quite simple,

$$K = (FT - FT')\,(S - S')$$

such that the root of the equation for the value of desired flight time in the Lambert's problem is given explicitly.

$$S = \frac{K}{(FT - FT')} + S'$$

*A Newtonian iteration could probably be used, but the equations for the derivatives of the flight time-S curves are very unwieldly and it is doubtful that this method would offer any advantage over the described curve fitting method.

The constants K, S', and FT', which usually have no realistic significance, are determined in a specific curve fit.

The primary advantage in using an equilateral hyperbola instead of, for example, a quadratic, is that it will always have only one root for any value of flight time. Thus, there is never any question as to which root to take, as indicated by the last equation. This makes the equilateral hyperbola quite ideal in describing the relationships between S (or e when $r_1 = r_2$) and both direct and indirect non-multiple orbit flight times, because in both cases there is only one value of S (or e) corresponding to any specific value of flight time, as can be seen in figure 12 (and 17).

A secondary advantage in the use of an equilateral hyperbola is that it can be readily fit to either three points, two points and an asymptote, or one point and two asymptotes.

It would seem that the equilateral hyperbola is unsuitable for dealing with multiple orbit flight times because for a specific value of flight time above the minimum, there will be two corresponding values of S (or e when $r_1 = r_2$) as can be seen in figure 16 (and 17). However, in any programmed logic, this minimum of a FT_{dN} or FT_{iN} curve should be defined before attempting a solution in order to ascertain that a solution exists; this minimum flight time must be less than that of the desired flight time. Once a solution is shown to exist, this minimum flight time point can be used as one of the points to which the hyperbola is fit in describing either "half" of the complete FT_{dN} or FT_{iN} curve, each half containing one of the two solutions. The "half" of the curve to which the hyperbola is fit will depend upon one of the specified characteristics of the solution sought as described at the very end of the previous section.

The following is a description of the hyperbola fit iterative procedure used by the author in finding a specific type of solution of a Lambert's problem. While the wording of this description is strictly applicable to the far more general category of problems wherein $r_1 \neq r_2$ (and the independent variable of iteration is S as shown in figure 16), the application of this method to the rather uncommon category of problems wherein $r_1 = r_2$ (and the independent variable is eccentricity as shown in figure 17) should be obvious.

1. Determine three consecutive members of the set of significant values of S (S1 to S7), the corresponding values of flight time of which bracket the desired value of flight time.
2. Fit an equilateral hyperbola to these three flight times-S conditions. Usually the three values of flight time at the three significant values of S will be finite, in which case the hyperbola is fit to three points. However, when an infinite value of flight time is involved, as is the case for direct flight time at S6, for indirect flight time at S2, and for all multiple orbit flight times at both S2 and S6, the value of S concerned is considered as an asymptote to the first hyperbola fit.
3. Calculate the root (value of S) of the hyperbola corresponding to the desired flight time.

4. Calculate the actual flight time corresponding to this value of S as given by the appropriate flight time equations (see appendix 10).

5. Fit a second hyperbola to the point derived from step 4 and the two points (or point and asymptote) of the previous hyperbola fit which are closest to the desired flight time. The flight times of the three points (or points and asymptote) should always bracket the desired flight time.

6. Take the root of this second hyperbola, calculate actual flight time, fit another hyperbola, etc. This iterative procedure should be terminated when the actual flight time calculated for a root is found to be exactly equal to or within a specified tolerance of the desired flight time, or whenever:

   a. No calculable change in flight time occurs for a change in S.

   b. The root of a hyperbola is exactly equal to the value of S used in a previous hyperbola fit.

   c. The root of a hyperbola is outside the range of S defined by the extreme values of S involved in the fit of the immediate hyperbola (remembering that these extreme values of S must bracket the desired value of S since the corresponding flight times bracket the desired flight time).

In all available experience with this method, most solutions are obtained in less than 6 iterations. This same experience indicates that this efficiency can be improved by using a modified hyperbola having the general form,

$$K = (FT - FT')\,(S - S')^{Q}$$

where Q would be given by an empirical equation as a function of R and A. This empirical equation has yet to be developed.

Appendix 1 - Equation for Eccentricity

Two positions at radii r_1 and r_2 on a conic separated by central angle A, must satisfy the following familiar equations which assume the conventions illustrated in figure 3.

$$r_1 = \frac{p}{1 + e\cos(S)} \qquad r_2 = \frac{p}{1 + e\cos(S+A)}$$

Eliminating (p) and combining,

$$r_1\,(1 + e\cos(S)) = r_2\,(1 + e\cos(S+A))$$

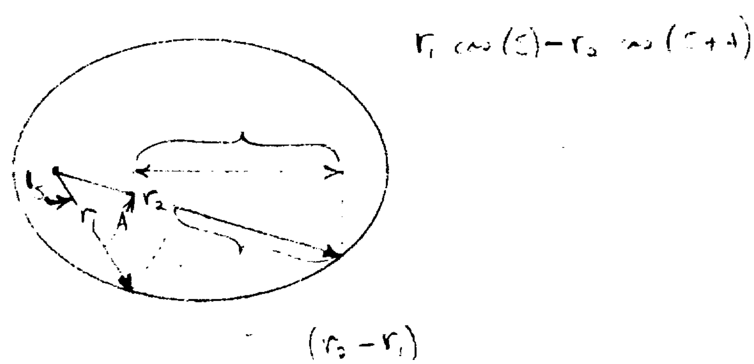
$$e\,(r_1\cos(S) - r_2\cos(S+A)) = r_2 - r_1$$

$$\therefore \quad e = \frac{r_2 - r_1}{r_1\cos(S) - r_2\cos(S+A)}$$

Or, making the substitution of $R = r_2/r_1$

$$e = \frac{R - 1}{\cos(S) - R\cos(S+A)}$$

A means of graphically illustrating the variation of eccentricity as a function of S can be derived from the equation immediately preceeding the substitution of R for r_2/r_1. It can be seen that the numerater of this equation $(r_2 - r_1)$ is the difference between the magnitude of the two vectors. The denominator $(r_1\cos(S) - r_2\cos(S+A))$ is the algebraic difference of the projections of r_1 and r_2 on the horizontal axis.



From this graphic model, values of S corresponding to extreme values of eccentricity could be derived for any vector set.

Appendix 2 - Eccentricity at S1

In the following equation for eccentricity derived in appendix 1,

$$e = \frac{r_2 - r_1}{r_1 \cos(S) - r_2 \cos(S+A)}$$

the denominator is actually the algebraic difference between the projections of r_1 and r_2 on the horizontal axis. Since by definition at S1 these two projections are equal in magnitude and sign, the denominator of the above equation will be zero and eccentricity will be infinite.

Appendix 3 - Expressions for S2 and S6

On a parabola, the two positions at r_1 and r_2 separated by central angle A must satisfy the following familiar equations,

$$r_1 = \frac{p}{1 + \cos(S)} \qquad\qquad r_2 = \frac{p}{1 + \cos(S+A)}$$

Eliminating (p) by combination and making the substitution of $R = r_2/r_1$

$$1 + \cos(S) = R\ (1 + \cos(S+A))$$

Expanding the cosine expression in the right member of this equation,

$$1 + \cos(S) = R\ (1 + \cos(S)\cos(A) - \sin(S)\sin(A))$$

Dividing by cos(S) and collecting terms,

$$1 - R\cos(A) + R\tan(S)\sin(A) = \frac{1}{\cos(S)}(R - 1) = (R - 1)\sec(S)$$

Substituting $\sqrt{\tan^2(S) + 1}$ for sec(S) and squaring,

$$(1 - R\cos(A))^2 + 2\,(1 - R\cos(A))\,R\tan(S)\sin(A) + (R\tan(S)\sin(A))^2 = (R - 1)^2\,(\tan^2(S) + 1)$$

Collecting terms yields a quadratic in tan(S)

$$\tan^2(S)\,(R^2\sin^2(A) - (R - 1)^2) + \tan(S)\,2R\sin(A)\,(1 - R\cos(A)) + (1 - R\cos(A))^2 - (R - 1)^2 = 0$$

Appendix 3 (con)

This quadratic can be simplified by making the following substitutions:

$X = R \sin(A) \qquad Y = 1 - R \cos(A) \qquad Z = R - 1$

$$\tan^2(S)\,(X^2 - Z^2) + \tan(S)\, 2\, X\, Y + Y^2 - Z^2$$

The two roots of this quadratic are thus given by the following:

$$\tan(S) = \frac{-2\,X\,Y \pm \sqrt{4\,X^2\,Y^2 - 4\,(X^2 - Z^2)\,(Y^2 - Z^2)}}{2\,(X^2 - Z^2)}$$

$$= \frac{-X\,Y \pm Z\sqrt{X^2 + Y^2 - Z^2}}{X^2 - Z^2}$$

Experience has shown that the positive sign option in this last equation always corresponds to a finite direct flight time, whereas the negative sign option always corresponds to an infinite direct flight time. Thus, by definition the positive sign option always yields S2 and the negative sign option always yields S6.

$$\tan(S2) = \frac{-X\,Y + Z\sqrt{X^2 + Y^2 - Z^2}}{X^2 - Z^2}$$

$$\tan(S6) = \frac{-X\,Y - Z\sqrt{X^2 + Y^2 - Z^2}}{X^2 - Z^2}$$

S6 will always be in either the 1st or 2nd quadrant and S2 will always be in either the 1st, 2nd, or 4th quadrants. In other words,

$$-90^\circ < S2 < 180^\circ \qquad 0 < S6 < 180^\circ \text{ where } \qquad S2 < S6$$

The preceeding expressions for S2 and S6 are presented in arc-tangent form for computer programming convenience. The denominator of these expressions are such that 90° values of both S2 and S6 will be given when $X^2 = Z^2$. This ambiguity does not exist in the following arc-cosine expression, the derivation of which is not given here because it is so similar to that of the arc-tangent expression.

Appendix 3 (Con)

$$S = \cos^{-1}\left(\frac{Y\ Z \pm X\sqrt{X^2 + Y^2 - Z^2}}{X^2 + Y^2}\right)$$

Substituting the equivalence $X = Z$ in this equation yields a 90° value of S when the negative sign option is taken, agreeing with the arctangent expression.

$$S = \cos^{-1}\left(\frac{X\ Y - X\ Y}{X^2 + Y^2}\right) = \cos^{-1}(0) = 90^\circ$$

However, taking the positive sign option yields a second root, which the arc-tangent expression does not.

$$S = \cos^{-1}\left(\frac{2\ X\ Y}{X^2 + Y^2}\right)$$

In order to express this second root in arc-tangent form, it is necessary to have it expressed in both arc-cosine and arc-sine form. The general arc-sine expression is as follows

$$S = \sin^{-1}\left(\frac{X\ Z \pm Y\sqrt{X^2 + Y^2 - Z^2}}{X^2 + Y^2}\right)$$

Substituting the equivalence $X = Z$ it can be seen that the positive sign option yields a 90° value of S, corresponding to the negative sign option in the arc-cosine expression.

$$S = \sin^{-1}\left(\frac{X^2 + Y^2}{X^2 + Y^2}\right) = \sin^{-1}(1) = 90^\circ$$

Combining the positive sign option arc-cosine expression and the negative sign option arc-sine expression yields an arc-tangent expression for the second root.

$$S = \tan^{-1}\left(\frac{\sin(S)}{\cos(S)}\right) = \tan^{-1}\left(\frac{X^2 - Y^2}{2\ X\ Y}\right)$$

The simple rules derived from experience determining which of S2 and S6 is 90° and which is given by the preceeding equation when $X^2 = Z^2$, and the quadrants in which they lie, are as follows:

when $A <^{*} 90^\circ$ $\quad S2 = 90^\circ \quad S6 = \tan^{-1}\left(\frac{X^2 - Y^2}{2\ X\ Y}\right)$, in 2nd quadrant

* Condition of equality not necessary to consider since when $A = 90^\circ$ X cannot equal Z, which is the situation under consideration.

Appendix 3 (Con)

when $90^{\circ} <^{*} A$ $\quad S6 = 90^{\circ}$ $\quad S2 = \tan^{-1}\left(\frac{X^2}{2} \frac{- Y^2}{X Y}\right)$ in 1st or 4th quadrants

The preceeding expression for the non-90° root of S when $X^2 = Z^2$ can also be derived from the quadratic equation used in the derivation of the general arc-tangent expressions for S2 and S6; the $\tan^2(S)$ term of this equation becomes zero when $X^2 = Z^2$ leaving a linear equation in tan(S).

The quadrant in which S6 lies, either 1st or 2nd, can be readily determined from the sign of the argument of its arc-tangent expression. The quadrant in which S2 lies, either 1st, 2nd, or 4th is determined by the signs of both the complete argument and the denominator $(X^2 - Z^2)$ of the argument of its arc-tangent expression, and the magnitude of A. Needless to say, if the complete argument is positive, S2 is in the 1st quadrant. If the complete argument is negative and A is greater than 90°, S2 is in the 4th quadrant. If the complete argument is negative and A is less than 90°, S2 will be in the 2nd quadrant if the denominator of the argument is negative, or S2 will be in the 4th quadrant if this denominator is positive.

The following bit of program logic in FORTRAN summarizes the calculation of S2 and S6 and the determination of the quadrants in which they lie for all possible cases, including cases where $X^2 = Z^2$. The arc-tangent function (ATAN) is assumed to yield a value in either the 1st or 4th quadrants, depending on the sign of its complete argument. It is also assumed that R, X, Y, and Z have been defined.

Appendix 3 (Con)

```
      DENOM = X * X - Z * Z
      CHAR = SQRT (X * X + Y * Y - Z * Z)
      C = 0.0
      IF (DENOM) 6, 3, 7
    3 DUMA = ATAN ((X * X - Y * Y) / (2.0 * X * Y))
      IF (A - PI/2.0) 4, 4, 5
    4 S2 = PI/2.0
      S6 = PI + DUMA
      GO TO 12
    5 S6 = PI/2.0
      S2 = DUMA
      GO TO 12
    6 C = PI
    7 S6 = ATAN (( -X * Y - Z * CHAR) / DENOM)
      IF (S6) 8, 9, 9
    8 S6 = PI + S6
    9 S2 = ATAN ((-X * Y + Z * CHAR) / DENOM)
      IF (S2) 10, 12, 12
   10 IF (A - PI/2.0) 11, 12, 12
   11 S2 = S2 + C
   12 CONTINUE
```

Appendix 4 - Expression for S3

The equation for eccentricity as derived in appendix 1 is

$$e = \frac{R - 1}{\cos(S) - R\cos(S+A)}$$

Taking the derivative of this expression with respect to S and setting it equal to zero,

$$\frac{de}{dS} = \frac{-(R-1)\ (-\sin(S) + R\sin(S+A))}{(\cos(S) - R\cos(S+A))^2} = 0$$

The value of S satisfying this equation is denoted as S3 by definition. In order for this equation to be satisfied, its numerator must equal zero, which requires the following since $R \neq 1$:

$$\sin(S3) = R\sin(S3+A)$$

Expanding the sine coefficient in the right member of this equation,

$$\sin(S3) = R\ (\sin(S3)\cos(A) + \sin(A)\cos(S3))$$

$$\sin(S3)\ (1 - R\cos(A)) = R\sin(A)\cos(S3)$$

$$\tan(S3) = \frac{\sin(S3)}{\cos(S3)} = \frac{R\sin(A)}{1 - R\cos(A)} = \frac{\sin(A)}{1/R - \cos(A)}$$

As pointed out in the text, S3 is that value of S such that the projections of r_1 and r_2 on an axis perpendicular to the periapsis horizontal, are equal in magnitude and coincident. This condition is stated explicitly in the third equation above.

Appendix 5 - Approximate expression for S5

S5, which corresponds to the ellipse having minimum period which can fit a vector set, must lie between S4 and S6. Experience shows that the period of the ellipse fitting a vector set is always decreasing at S4 as S is increasing. At S6, the period goes to an infinite limit, the conic fitting the vector set having become a parabola.

Appendix 5 (Con)

It would thus seem that a logical form of an empirical expression for S5 would be as follows:

$$S5 = S4 + (S6 - S4)\,f$$

where the fractional coefficient (f) of the difference between S6 and S4 is expressed as an empirical function of the values of R and A of the vector set.

However, experience has shown that this form does not work as well as the following form,

$$S5 = S4 + (S7 - S4)\,f$$

This form would not seem to be logical in that S7, which is the implied upper limit of S5 when f=1 in this form, is outside the range of elliptical conics with which we are concerned. The range of from S6 to S7 is associated only with non-multiple orbit indirect trajectories, usually having very short flight times. Since both S4 and S7 are simple explicit functions of A, this form of empirical equation for S5 can be restated as,

$$S5 = 180^\circ - A + \frac{A\,f}{2}$$

A computer program was used to iteratively derive the ellipse having the minimum period which fits a given vector set (and hence, to derive S5). Values of S5 were thus calculated for numerous vector sets having different combinations of R and A over wide ranges of their values. The following empirical equation, having the form past described, was then derived to express these calculated values of S5 as functions of R and A.

$$S5 = 180^\circ - A + \frac{A}{2}\left((\sin(A))^{1.5(1-1/(2R-1))}\, e^{-3.43(\sqrt{R}-1)}\right)$$

In most instances, the value of S5 as given by this equation is within a degree of arc of the actual value.

Appendix 6 - Eccentricity and flight time at S7

The following equation, derived in appendix 1, gives the eccentricity of any conic fitting a vector set corresponding to any value of S.

$$e = \frac{R - 1}{\cos(S) - R\cos(S+A)}$$

Substituting the value of S7, which is $180^\circ - A/2$, into this equation,

$$e_{S7} = \frac{R - 1}{\cos(180^\circ - A/2) - R\cos(180^\circ - A/2 + A)}$$

$$= \frac{R - 1}{\cos(180^\circ - A/2) - R\cos(180^\circ + A/2)}$$

Since $\cos(180^\circ - A/2) = \cos(180^\circ + A/2)$ and $\cos(180^\circ - A/2) = -\cos(A/2)$,

$$e_{S7} = \frac{R - 1}{(R - 1)\cos(A/2)} = \sec(A/2)$$

At S7, the cosine of the true anomalies of both r_1 and r_2 is equal to $-\cos(A/2)$. Substituting this value for $\cos(S)$ in the familiar positional equation as applied to r_1,

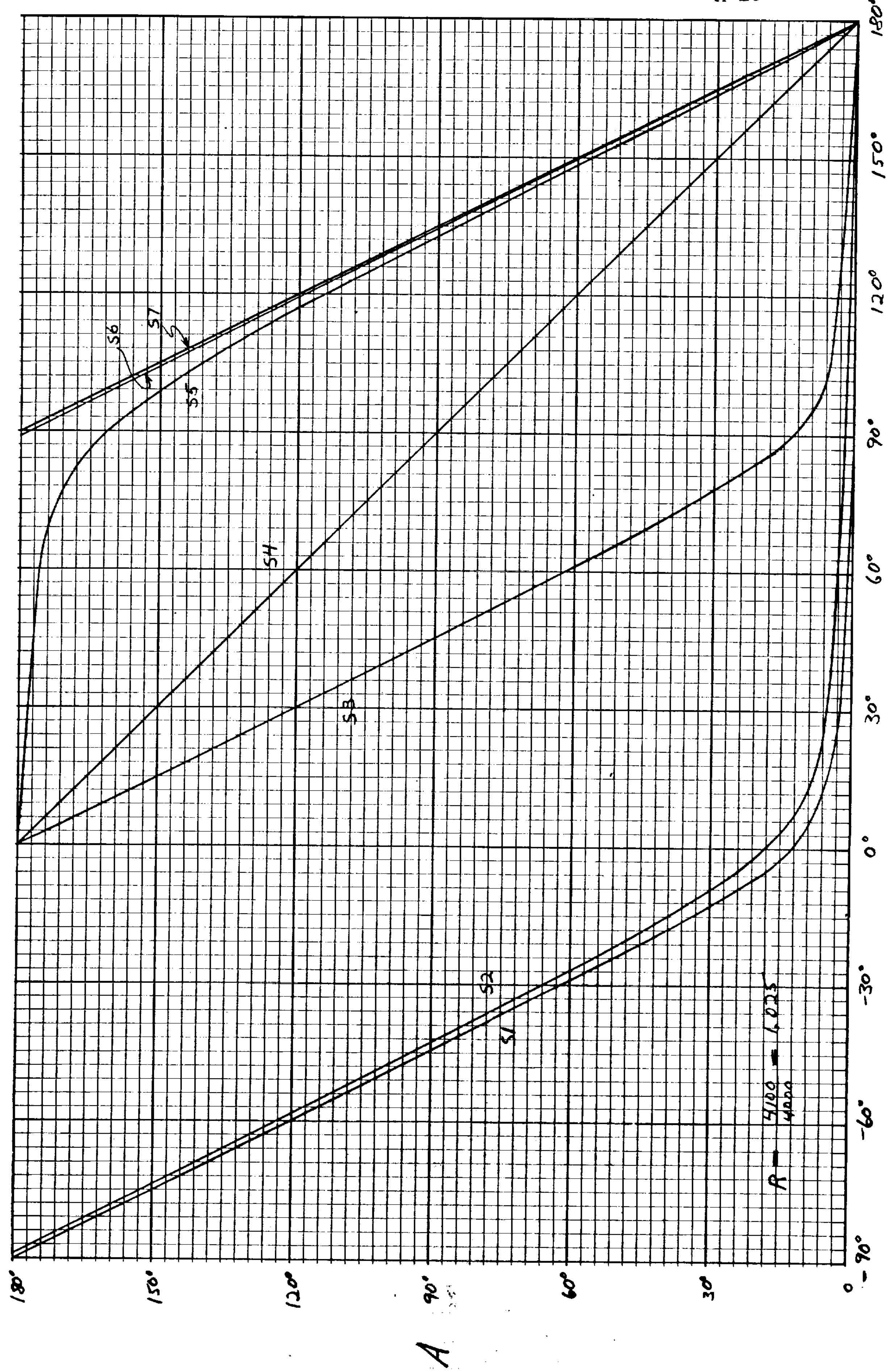
$$r_1 = \frac{p}{1 + e\cos(S)} = \frac{p}{1 - \cos(A/2)\sec(A/2)} = \frac{p}{1 - 1} = \frac{p}{0}$$

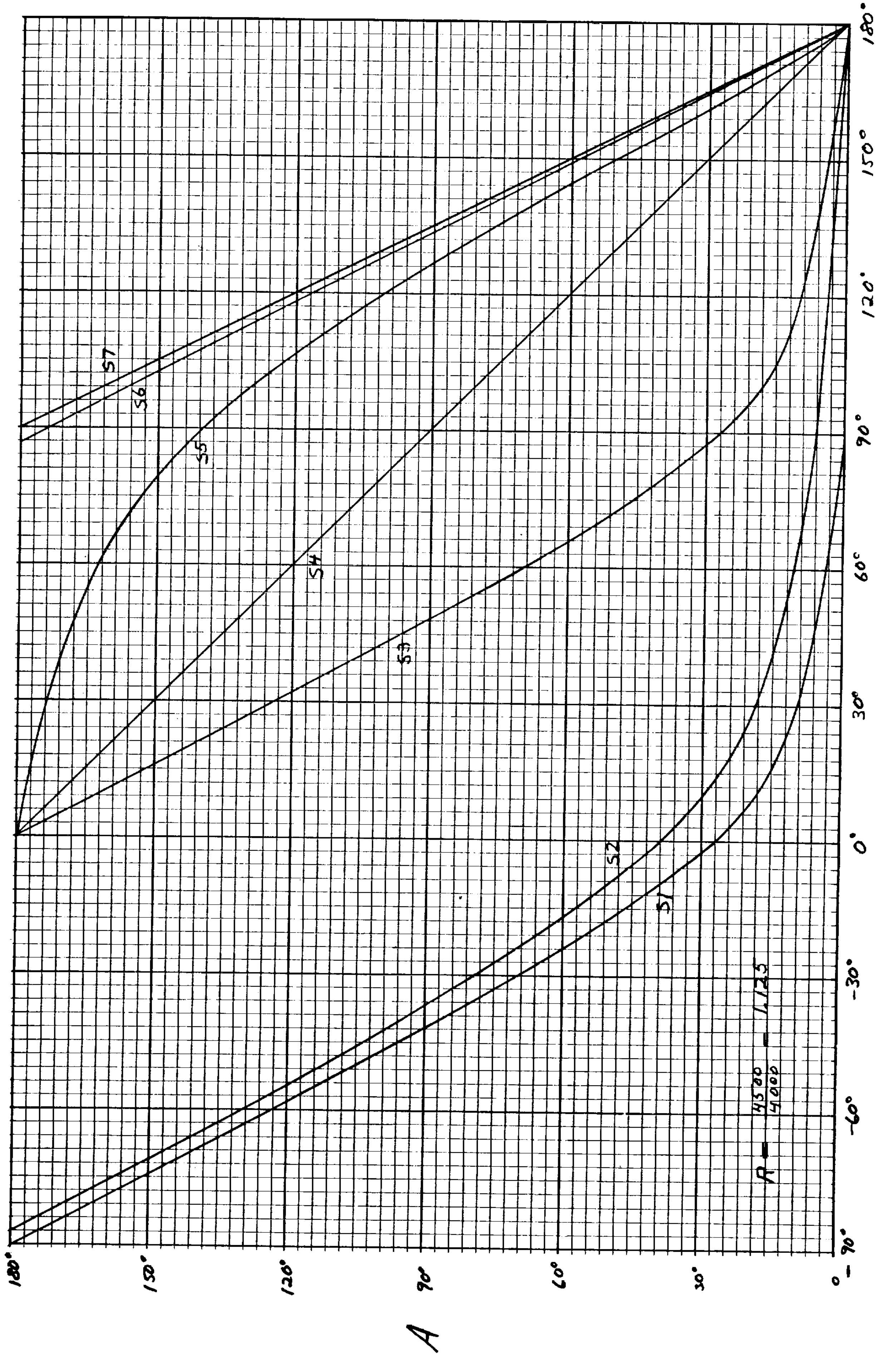
The result is that since r_1 and r_2 are known to be finite, the semi-latus rectum (p) of the hyperbola fitting a vector set at S7 must be zero. This S7 hyperbola is actually coincident with r_1 and r_2 and passes through the center of attraction.

In appendix 10, it is shown that the coefficient of all equations expressing flight time from periapsis as functions of true anomaly, radius, and eccentricity, is

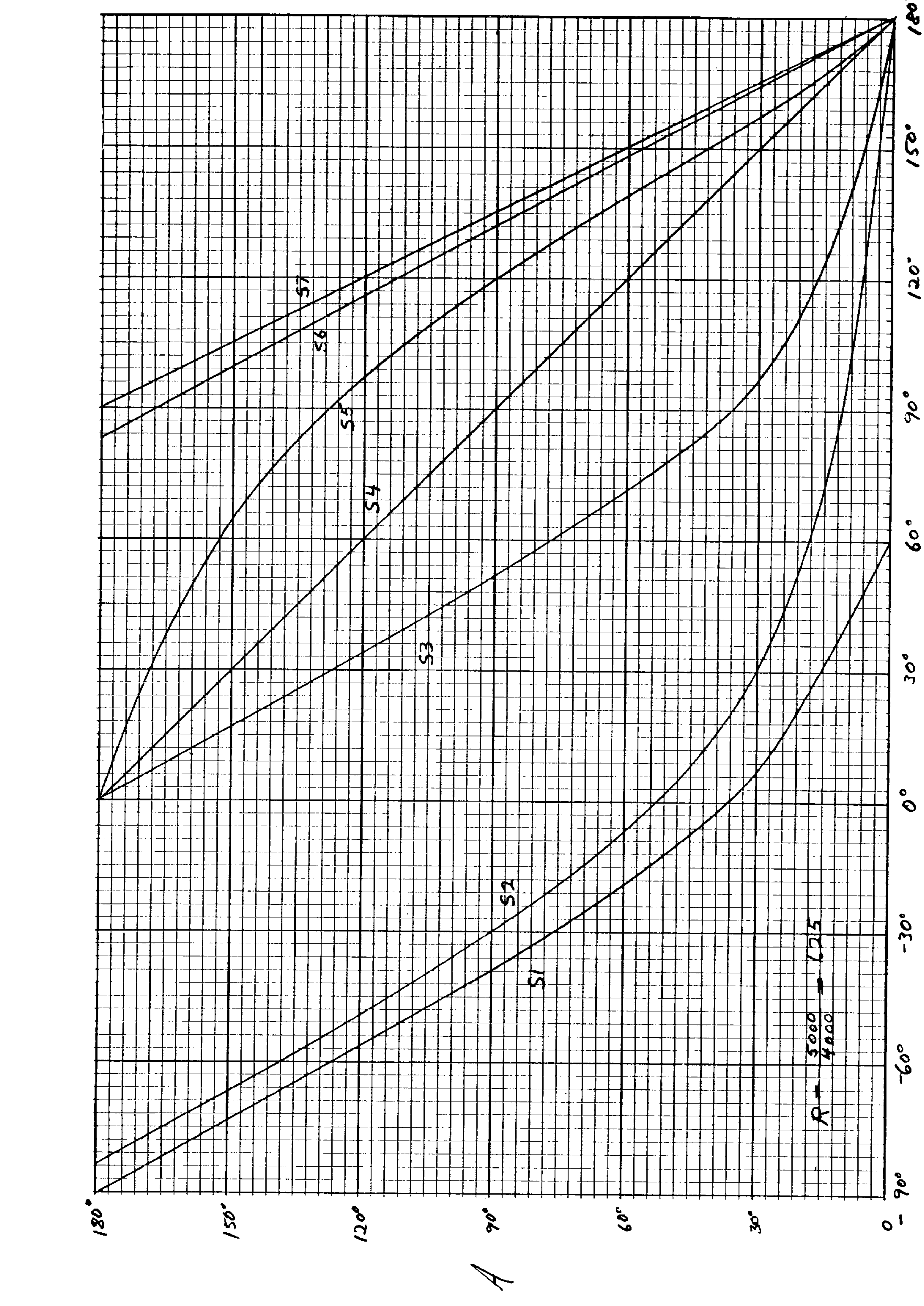
$$C^3/\mu^2, \text{ which is equal to } p^{3/2}/\mu^{1/2}.$$

Since p is zero at S7, this coefficient of the aforementioned hyperbolic flight time equation will be zero, resulting in a zero flight time.

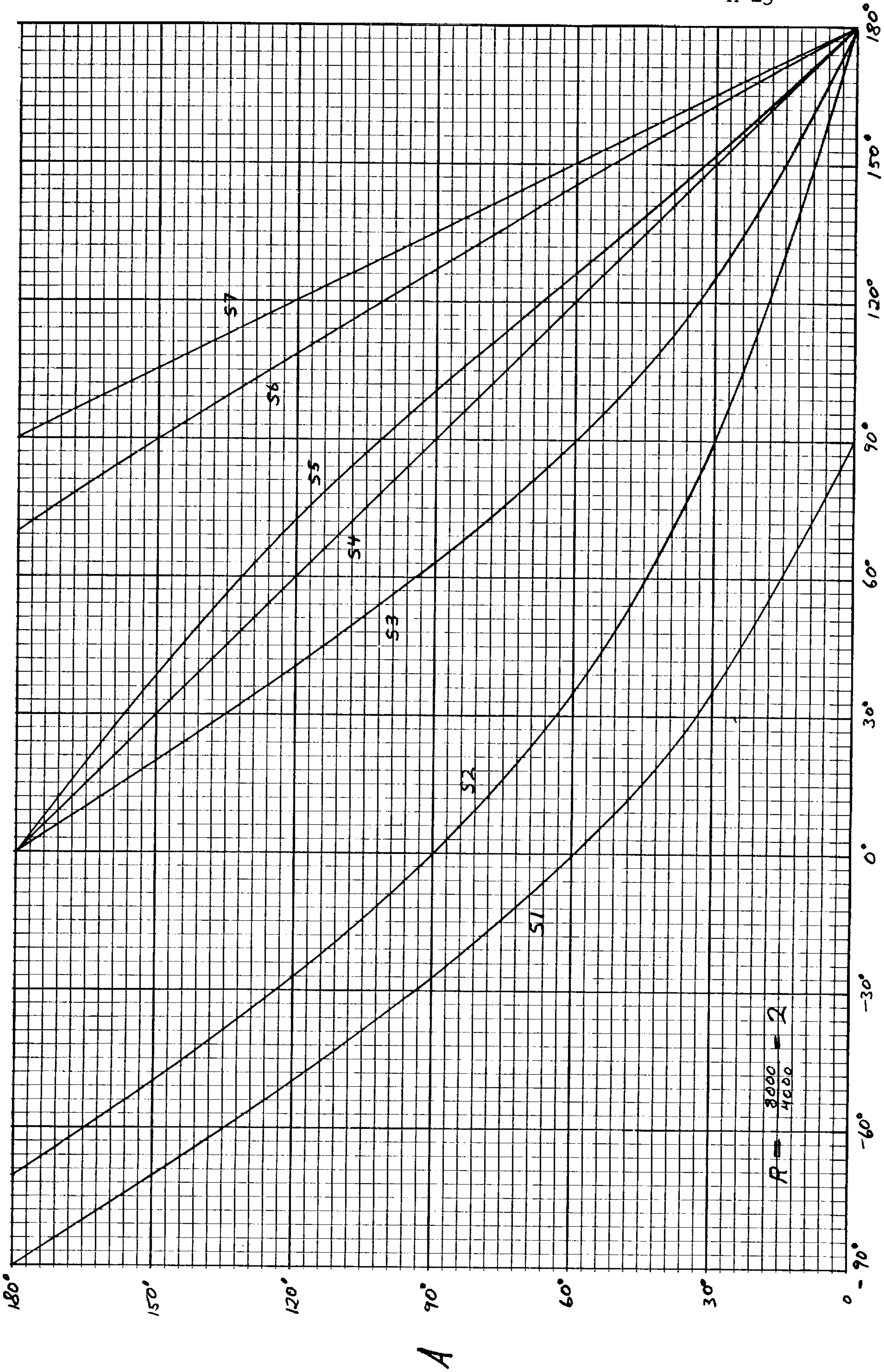

S1
S2
S3
S4
S5
S6
S7
R = 4100/4000 = 1.025
A
180°
150°
120°
90°
60°
30°
0°
-30°
-60°
-90°
0

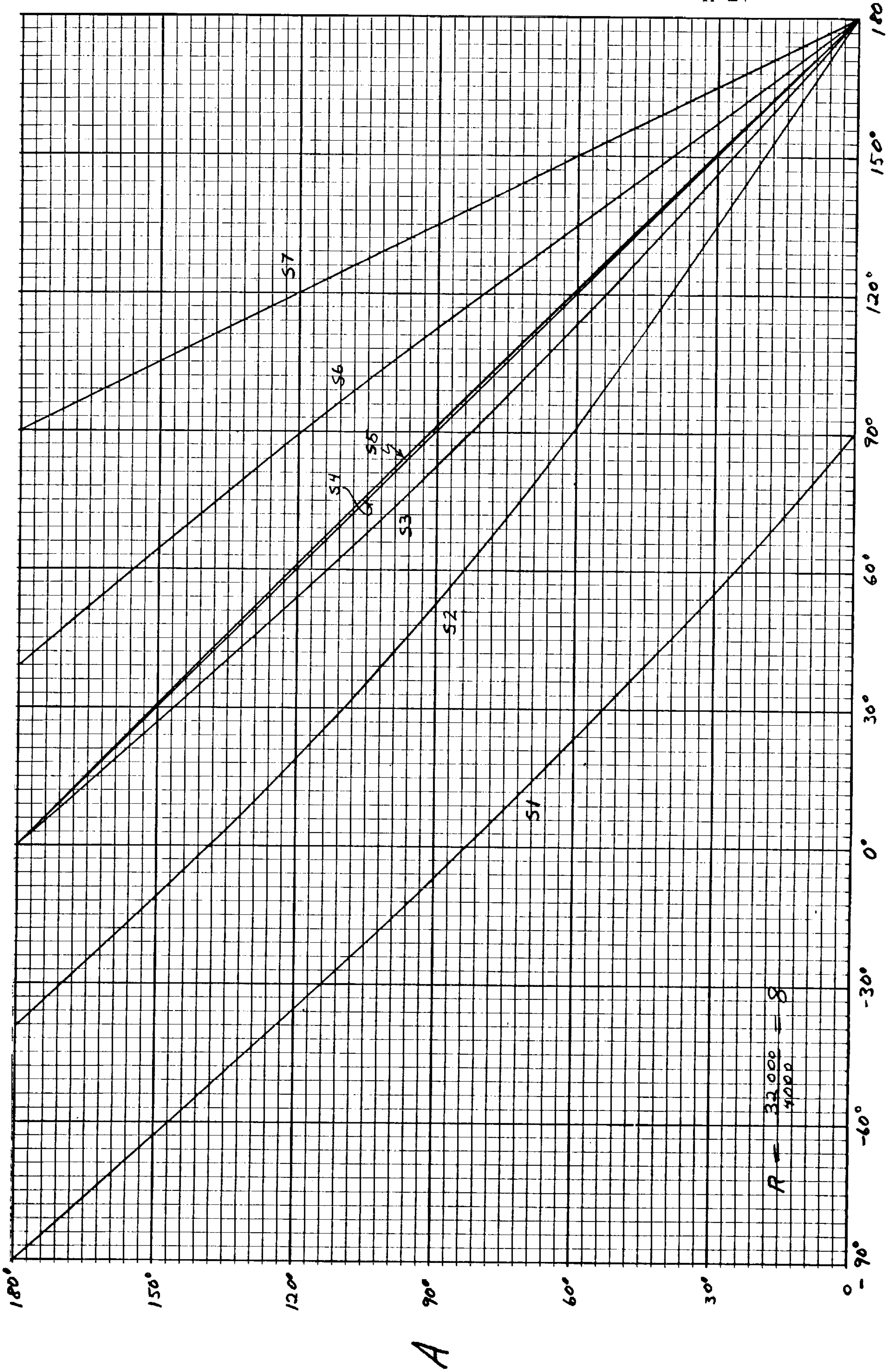


A
$R = \frac{4500}{4000} = 1.125$
51
52
53
54
55
56
57
180°
150°
120°
90°
60°
30°
0
-30°
-60°
-90°



S1
S2
S3
S4
S5
S6
S7
$R = \frac{5000}{4000} = 1.25$
180°
150°
120°
90°
60°
30°
0°
-30°
-60°
-90°
0
A


S1
S2
S3
S4
S5
S6
S7
R = 8000/4000 = 2
180°
150°
120°
90°
60°
30°
0°
-30°
-60°
-90°
180°
150°
120°
90°
60°
30°
0
A


S7
S6
S5
S4
S3
S2
S1
180°
150°
120°
90°
60°
30°
0°
-30°
-60°
-90°
0
A
R = 32,000/4000 = 8

Appendix 7 - Variation of significant values of S as functions of R and A.

The preceding five figures illustrate the dependence of the significant values of S1 to S7 on the values of R and A of vector sets. Each figure represents a fixed value of R and each curve in a figure represents one of the seven significant values of S. The ordinate of each figure represents all possible values of A of from 0° to 180°. The abscissa of each figure represents the three quadrants of from -90° to 180^{0} within which values of S exist. As projected on this abscissa scale, the intersections of a horizontal line, corresponding to a given value of A, with the seven curves, will represent the magnitudes of the seven significant values of S.

Appendix 8 - Invariance of (p) when A = 180^{0}

For any conic fitting a vector set, the following two positional equations must be simultaneously satisfied:

$$r_1 = \frac{p}{1 + e\cos(S)} \qquad\qquad r_2 = \frac{p}{1 + e\cos(S+A)}$$

Combining these two equations by eliminating (p), and substituting the condition of $A = 180^{0}$,

$$r_1\ (1 + e\cos(S)) = r_2\ (1 + e\cos(S + 180^{\circ})) = r_2\ (1 - e\cos(S))$$

From this equation, the following expression for (e) is derived,

$$e_{(A=180^{\circ})} = \frac{1}{\cos(S)}\left(\frac{r_2 - r_1}{r_1 + r_2}\right)$$

Taking the derivatives of this expression with respect to S,

$$\frac{de}{dS}\,(A=180^{\circ}) = \frac{\sin(S)}{\cos^2(S)}\left(\frac{r_2 - r_1}{r_1 + r_2}\right)$$

The following expression for the derivative of (p) with respect to S is derived from the first equation above:

$$\frac{dp}{dS} = r_1\ \left(\cos(S)\,\frac{de}{dS} - e\sin(S)\right)$$

Substituting the expressions for (e) and de/dS when $A=180^{\circ}$ in this equation, the value of the equation is found to be zero. This indicates that when $A=180^{0}$, (p) is a function of r_1 and r_2 and is independent of S.

Appendix 8 (Con)

$$\frac{dp}{dS}\,(A=180^{\circ}) = r_1\left[\frac{\cos(S)\,\sin(S)}{\cos^2(S)}\left(\frac{r_2 - r_1}{r_1 + r_2}\right) - \frac{\sin(S)}{\cos(S)}\left(\frac{r_2 - r_1}{r_1 + r_2}\right)\right] = 0$$

Substituting the expression for eccentricity when $A=180^{\circ}$ in the first equation gives the following expression for (p):

$$p_{(A=180^{\circ})} = r_1\left(1 + \frac{r_2 - r_1}{r_2 + r_1}\right) = \frac{2\,r_1\,r_2}{r_1 + r_2}$$

Appendix 9 - Expression for e_{minp}

The following familiar conic equations must be satisfied by any conic fitting any vector set:

$$r_1 = \frac{p}{1 + e\cos(S)} \qquad\qquad a = \frac{p}{1 - e^2}$$

Eliminating (p) from these equations yields the following equation for (a)

$$a = \frac{r_1\,(1 + e\cos(S))}{1 - e^2}$$

When $r_1 = r_2$, the value of S is a constant in any Lambert's problem. The derivative of (a) with respect to eccentricity is thus,

$$\frac{da}{de} = \frac{(1 - e^2)\,r\cos(S) - r\,(1 + e\cos(S))\,(-2)\,e}{(1 - e^2)^2}$$

Equating this derivative to zero and solving for its root,

$$(1 - e^2)\cos(S) + 2\,e\,(1 + e\cos(S)) = 0$$

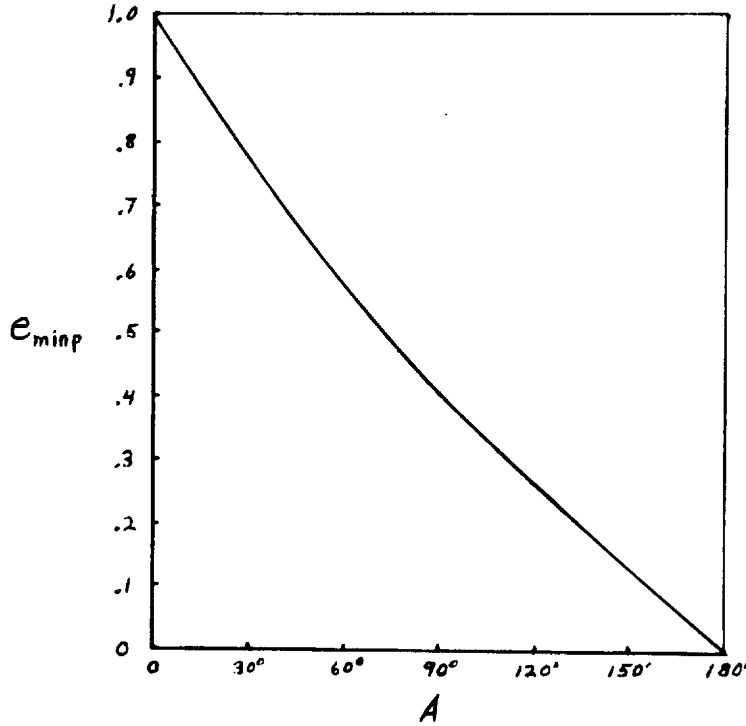
$$e^2\cos(S) + 2\,e + \cos(S) = 0$$

$$e_{minp} = \frac{-2 \pm \sqrt{4 - 4\cos^2(S)}}{2\cos(S)} = \frac{-1 \pm \sin(S)}{\cos(S)}$$

The table on page 28 shows that of the two possible values of S in any problem wherein $r_1 = r_2$, the magnitude of S corresponding to the minimum period ellipse is $\pm(180^{\circ} - A/2)$. Substituting this value for S and taking the appropriate sign option such that e_{minp} is always positive,

$$e_{minp} = \frac{1 - \sin(A/2)}{\cos(A/2)} = \sec(A/2) - \tan(A/2)$$

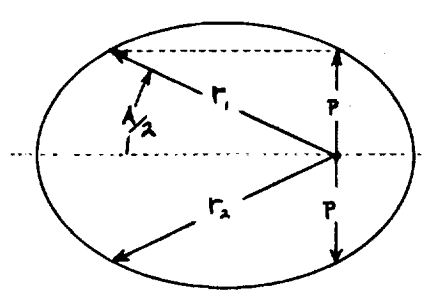
The following graph shows the variation of e_{minp} as a function of A.



Substituting the expression for e_{minp} and the value of $(180^\circ - A/2)$ for S in the first equation, yields the following equation for (p):

$$p = r\ (1 + e \cos(S)) = r\ (1 - \cos(A/2)\ (\sec(A/2) - \tan(A/2)))$$
$$= r\ (1 - 1 + \sin(A/2)) = r \sin(A/2)$$

This relationship indicates that on the minimum period ellipse fitting a vector set wherein $r_1 = r_2$, the projection of either r_1 or r_2 on a parallel to the minor axis is equal to (p).



Appendix 10 - Flight time equations and series

Flight time (FT), expressed as a function of true anomaly (θ), between two positions on a conic trajectory having true anomalies θ_1 and θ_2, is given by the following integral expression;

$$FT = \frac{C^3}{\mu^2} \int_{\theta_1}^{\theta_2} \frac{d\theta}{(1 + e\cos(\theta))^2}$$

where C is the angular momentum equal to ($r\ v\cos(\gamma)$) at any position on the conic, and μ is the gravitational constant of the central force field.

There is no single equation expressing the integral in the above equation for all values of eccentricity (e). Three different equations must be used, each being applicable to values of eccentricity which are either less than unity, equal to unity, or greater than unity. The resulting flight time equations are referred to respectively as elliptical, parabolic, and hyperbolic.

As conventionally stated in their simplest form, these flight time equations express flight time from periapsis to a position having true anomaly (θ). These equations are thus solutions of the above integral equation wherein $\theta_1=0$. The flight time equations presented herein are of this form.

There are numerous ways of expressing these three flight time equations, each involving different combinations of different orbital elements of the conic trajectory. Because in the method of solving Lambert's problem with which this paper is concerned the only orbital element necessarily involved is eccentricity, the flight time equations presented here do not involve any orbital elements other than eccentricity. These three flight time equations, thus unconventionally expressed, might appear unfamiliar and cumbersome compared to the more conventional forms involving additional orbital elements.

The following are alternate expressions for the coefficient of the integral in the first equation,

Appendix 10 (Con)

$$\frac{C^3}{\mu^2} = \frac{p^2}{C} = \frac{r^2\,(1 + e\cos(\theta))^2}{\sqrt{p\,\mu}} = \frac{(r\,(1 + e\cos(\theta)))^{3/2}}{\sqrt{\mu}}$$

the last expression involving no orbital elements other than eccentricity. In each flight time equation, this coefficient will be denoted simply as C^3/μ^2 and the integral will be expressed as a function of θ and (e). The units of this coefficient are time whereas the integral expressions are dimensionless.

In the following flight time equations, the arctangent function is assumed to yield values in the first or fourth quadrants, depending upon the sign of its complete argument. Thus, flight time as given by these equations will be positive when θ is in the first or second quadrants, and negative when θ is in the third or fourth quadrants.

Elliptical flight time ($e < 1$)

$$FT = \frac{C^3}{\mu^2}\left[\frac{2}{(1 - e^2)^{3/2}} \tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\ \tan(\theta/2)\right) - \frac{e\sin(\theta)}{(1 - e^2)\,(1 + e\cos(\theta))}\right]$$

Parabolic flight time ($e = 1$)

$$FT = \frac{C^3}{\mu^2}\quad \frac{\tan(\theta/2)}{2}\left(1 + \frac{\tan^2(\theta/2)}{3}\right)$$

or,

$$FT = \frac{C^3}{\mu^2}\quad \frac{\tan(\theta/2)}{3}\left(\frac{2 + \cos(\theta)}{1 + \cos(\theta)}\right)$$

Hyperbolic flight time ($e > 1$)

$$FT = \frac{C^3}{\mu^2}\left[\frac{e\sin(\theta)}{(e^2-1)\,(1+e\cos(\theta))} - \frac{1}{(e^2-1)^{3/2}}\ \ln\left(\frac{\sqrt{(e+1)/(e-1)} + \tan(\theta/2)}{\sqrt{(e+1)/(e-1)} - \tan(\theta/2)}\right)\right]$$

The most commonly used flight time equation will be the elliptical. This equation is used for all values of S between S2 and S6. The period of the ellipses within this range, used in calculating multiple orbit flight times, is given by,

Appendix 10 (Con)

$$P = \frac{C^3}{\mu^2} \frac{2\pi}{(1 - e^2)^{3/2}}$$

The hyperbolic flight time equation is used only for values of S between S1 and S2 and between S6 and S7. The parabolic flight time equation represents the limit which both the elliptical and hyperbolic equations approach as S approaches S2 or S6 and eccentricity approaches unity. In the iterative curve-fit logic described in the last section of this paper, the parabolic flight time equation would only be used to calculate either the direct flight time at S2 or indirect flight time at S6 for the initial curve fit of a problem.

Difficulty is sometimes encountered when the desired non-multiple orbit flight time in a given Lambert's problem is very close to, but not exactly equal to, the finite parabolic flight time. This difficulty is due to the erratic behavior of both elliptical and hyperbolic flight time equations in actual calculations when eccentricity is very close to unity. In these cases, the calculated values of flight time do not vary continuously in approaching the finite parabolic flight time as eccentricity approaches unity. This difficulty is reduced by using more significant figures in the calculations. Needless to say, if one could use an infinite number of significant figures in using these flight time equations, this difficulty would not exist.

When the condition is satisfied that,

$$\left(\frac{1 - e}{1 + e} \tan^2(\theta/2)\right) < 1$$

it is possible to use the following series expression for flight time past periapsis:

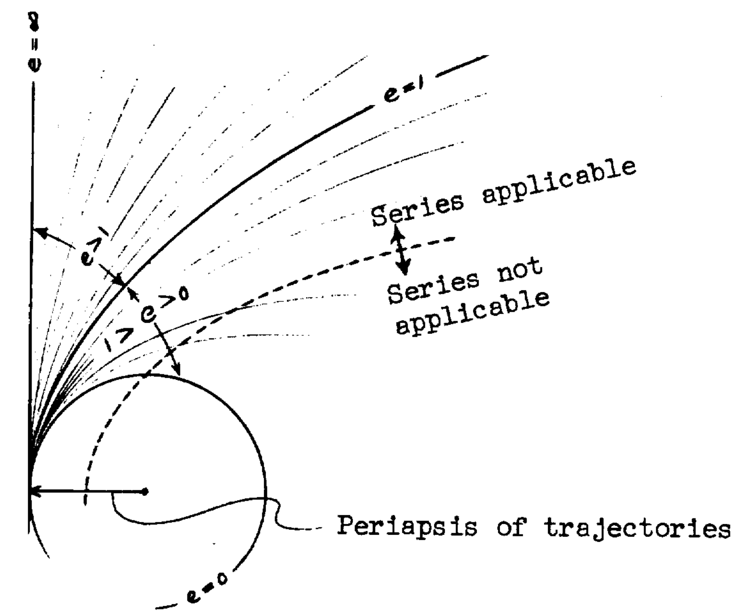
$$FT = \frac{C^3}{\mu^2} 2 \left[\frac{(2+e(1+\cos(\theta)))\tan(\theta/2)}{2(1+e\cos(\theta))(1+e)^2} - \frac{\tan^3(\theta/2)}{3(1+e)^3} + \frac{(1-e)\tan^5(\theta/2)}{5(1+e)^4} \cdots\cdots \right]$$

While this series expression is not as convenient to use as the three explicit equations, it has the advantage that it will give elliptical, parabolic, and hyperbolic flight times. This series expression is

Appendix 10 (Con)

well-behaved and continuous across the parabolic condition and thus offers a means of dealing with the discontinuity difficulty just described.

It can be seen that this series will converge whenever eccentricity is greater or equal to unity, regardless of the true anomaly. When eccentricity is less than unity, convergence will be a function of θ. Thus, the series expression can be used for any hyperbolic or parabolic trajectory but is restricted in application to elliptical trajectories. The following figure graphically illustrates this restriction.



The series expression can be used for positions on all conics which are "outside" a parabolic boundary, the periapsis of which is one half the magnitude of and coincident with that of the conic.